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Slicing Skew-tableau Frames

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Using special decompositions of the frame into zigzag paths, a simple algorithm is given for reducing the enumeration of skew-tableaux to the enumeration of k-tuples of linear partitions. This gives a direct combinatorial method for obtaining the homomorphic image of a frame in the ring $Z[x_1, x_2, \ldots]$ of polynomials in an infinite sequence of independent indeterminates x_i and thus gives a combinatorial interpretation to the non-zero terms that arise from the classical expression for a skew Schur function as a determinant of homogeneous symmetric functions (or of elementary symmetric functions). In the special case of tableau frames, it provides direct combinatorial significance to the signed Kostka numbers of the second kind.

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The use of zigzag paths here is similar to that in the authors "peeling" proof in [2] of Stanley's hook number generating function for reverse plane partitions. The main results generalize theorems of Foulkes in [1, Section 6] on skew-hooks to general skew-tableau frames and their arenas.

Examples are given in Section 12.

1. Frames and Arenas

If $\alpha = \langle a_1, \ldots, a_d \rangle$ with the a_i integers satisfying $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_d \geqslant 0$, α is a *partition* of *size* $|\alpha| = a_1 + \cdots + a_d$ and the set of *nodes*

$$F(\alpha) = \{(p, q) : p = 1, 2, \ldots, d; q = 1, 2, \ldots, a_p\}$$

is the *tableau frame* of shape α and *size* $|F(\alpha)| = |\alpha|$. If $F(\beta)$ is a subset of $F(\alpha)$, the (skew-tableau) *frame* $F(\alpha/\beta)$ consists of the nodes that are in $F(\alpha)$ and are not in $F(\beta)$. If $|\beta| = 0$, $F(\alpha/\beta) = F(\alpha)$. The *size* of $F = F(\alpha/\beta)$ is the number $|F(\alpha/\beta)| = |\alpha| - |\beta|$ of nodes in F. The *standard indexing* of the nodes (p_j, q_j) of a frame F is such that, for $1 \leqslant j \leqslant |F|$,

$$\text{either} \quad p_{j+1} = p_j \quad \text{and} \quad q_{j+1} = 1 + q_j \quad \text{or} \quad p_{j+1} = p_j - 1. \tag{1.1}$$

The components p and q are the *row number* and *column number*, respectively, of the node (p, q). The *overlap* for a frame F is the set of row numbers p for which there exists

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a column number q such that both (p, q) and $(p+1, q)$ are in F. A *gate* for F is a subset of the overlap for F; an *arena* is an ordered pair (F, G) with G a gate for F.

If F is a frame, its *conjugate* (or *transpose*) is the frame $F^{\mathrm{T}} = \{(q, p): (p, q) \in F\}$. For every positive integer s, let $L(s)$ denote the one-rowed frame $F(\langle s \rangle) = \{(1, j): 1 \leq j \leq s\}$ of size s; also let x_s be the arena $(L(s), \varnothing)$ and y_s be the arena $([L(s)]^{\mathrm{T}}, \varnothing)$.

2. Maps

Let $N = \{0, 1, \ldots\}$. A *measure* $M = \{m_n\}$ is a sequence $m_0, m_1, \ldots$ with each m_n in N and with $m_n > 0$ for only a finite set of n. The *size* $|M|$ of a measure M is $\sum m_n$ and the *weight* $\|M\|$ is $\sum nm_n$, where each sum is over all n with $m_n > 0$.

Let R be a mapping from a frame F into N. The *measure* μR is the sequence $\{m_n\}$ such that $R(p, q) = n$ for exactly m_n nodes (p, q) of F and the *weight* $\|R\| = \|\mu R\|$. We note that the size $|\mu R| = |F|$ and the weight $\|R\| = \sum R(p, q)$, summed over all nodes (p, q) of F. The *gate* γR is the set of all p for which (p, q) and $(p+1, q)$ exist in F with $R(p, q) \geq R(p+1, q)$. We say that R is a row-monotonic map (*rm-map*) on F if $R(p, q)$ is a non-decreasing function of q for each fixed p. R is a *map on an arena* (F, G) if R is an *rm*-map on F and $\gamma R = G$. Thus a map on $(F, \varnothing)$ is a skew-tableau on F.

If $A = (F, G)$ is an arena and M is a measure, the number of maps R on A with $\mu R = M$ will be denoted by $\nu[A, M]$ or by $\nu(F, G, M)$. Clearly, $\nu(F, G, M) = 0$ if $|F| \neq |M|$.

3. Sums and Products

The sum $\{m'_n\} + \{m''_n\}$ of two measures is the sequence $\{m_n\}$ with $m_n = m'_n + m''_n$ for all n in N.

Let F and F' be frames with e as the largest column number q for the nodes (p, q) of F and d as the largest row number p' for the (p', q') of F'. We define the *frame product* of F and F' (in that order) to be

$$FF' = \{(p+d, q): (p, q) \in F\} \cup \{(p', q'+e): (p', q') \in F'\}.$$

The *arena product* $(F, G)(F', G')$ is the (FF', G'') in which G'' is the gate with p in G'' if p is in G' or $p-d$ is in G. Clearly, arena multiplication is associative.

The *mapping product* of mappings U and V from F and F', respectively, into N is the mapping $R = UV$ on FF' given by

$$R(p+d, q) = U(p, q) \quad \text{for } (p, q) \in F,$$
$$R(p', q'+e) = V(p', q') \quad \text{for } (p', q') \in F'.$$

Clearly, every mapping R from FF' to N is such a product UV and $\mu(UV) = \mu(U) + \mu(V)$. Also, if U and V are maps on arenas (F, G) and (F', G'), respectively, UV is a map on their product $(F, G)(F', G')$.

If A_1 and A_2 are arenas and M is a measure, one sees easily that

$$\nu[A_1A_2, M] = \sum \nu[A_1, M_1]\nu[A_2, M_2], \tag{3.1}$$

where the sum is over all ordered pairs (M_1, M_2) with $M_1 + M_2 = M$.

Let $M = m_0, m_1, \ldots$ be a measure with $i, j, \ldots, k$ as the values of n for which $m_n > 0$; then let $X[M] = L(m_i)L(m_j) \cdots L(m_k)$.

4. The Rings Ω and Λ

Let Ω consist of all finite linear combinations $C = u_1A_1 + \cdots + u_hA_h$ of arenas A_i with integral coefficients u_i. In Ω, let addition be by collecting like terms and multiplication be the extension of arena multiplication by distributivity, i.e.

$$\left(\sum_{i=1}^{h} u_i A_i\right)\left(\sum_{j=1}^{k} v_j B_j\right) = \sum_{i=1}^{h} \sum_{j=1}^{k} u_i v_j (A_i B_j).$$

This makes Ω into a non-commutative ring with the arena $(\varnothing, \varnothing)$ as its unity. We extend the definition of $\nu[A, M]$ by letting

$$\nu\left[\sum_{i=1}^{h} u_i A_i, M\right] = \sum_{i=1}^{h} u_i \nu[A_i, M]. \tag{4.1}$$

Also we consider Ω to contain all linear combinations of frames by letting a frame F represent the arena $(F, \varnothing)$.

For C and D in Ω, we write $C \sim D$ to mean that $\nu[C, M] = \nu[D, M]$ for all M. Let I be the subset of C in Ω such that $\nu[C, M] = 0$ for all measures M. Then it follows from (3.1) and (4.1) that I is a two-sided ideal in Ω and we let Λ be the quotient ring Ω/I. Thus $C \sim D$ in Ω becomes $C = D$ in Λ. In Section 9 we show that the present Λ is isomorphic to the ring $Z[x_1, x_2, \ldots]$ of polynomials, with integer coefficients, in an infinite sequence of independent indeterminates x_i and hence is isomorphic to the Λ developed in [3] and [4]. Clearly, our definition of $\sim$ implies that Λ is a commutative ring.

5. Paths and Slices

Let $F = F(\alpha/\beta)$, $\alpha = \langle a_1, \ldots, a_d \rangle$, $\beta = \langle b_1, \ldots, b_d \rangle$, and $b_d < a_d$. If there exist positive integers $c_{d+1}, c_d, \ldots, c_h$ such that

$$b_{i-1} < c_i \leq a_i \quad \text{for } h < i \leq d \quad \text{and} \quad 1 + b_d = c_{d+1} \leq c_d \leq \cdots \leq c_h = a_h,$$

then the set

$$\pi(c_{d+1}, c_d, \ldots, c_h) = \{(p, q) \colon h \leq p \leq d,\ c_{p+1} \leq q \leq c_p\}$$

is a subset of F and we call it an *h-path* in F. If $b_{i-1} < a_i$ for $h < i \leq d$, $\pi(1 + b_d, a_d, a_{d-1}, \ldots, a_h)$ is called the *rimpath* ρ_h and the complement of ρ_h in F is denoted by E_h. Each of ρ_h and E_h is a subframe of F.

A *slicing* of F is a partitioning of F into subsets $\pi_1, \ldots, \pi_k$ such that π_1 is one of the rimpaths ρ_h of F and π_i for $i > 1$ is one of the rimpaths of the frame resulting from the deletion of the nodes of $\pi_1, \ldots, \pi_{i-1}$ from F. Since each rimpath of a frame contains the entire last row of the frame, the number of *slices* π_i in a slicing of F is at most the number of rows with nodes in F. It is also easy to show inductively that the number of slicings of F is at most $d!$.

A *slicing for an arena* (F, G) is a slicing $\pi_1, \ldots, \pi_k$ of F such that whenever p is in G there exist q and j for which (p, q) and $(p + 1, q)$ are both nodes of the same slice π_j.

6. Routes and Streaks

Let $F = F(\alpha/\beta)$, α, β, d, the a_i, and the b_i be as in Section 5. Let the rimpaths of F be $\rho_d, \rho_{d-1}, \ldots, \rho_r$ and $s_i = |\rho_i|$. One sees that $s_i = d + a_i - b_d - i$ for $r \leq i \leq d$. For each *rm*-map R on F, we define inductively the c_i for a path $\pi[R] = \pi(c_{d+1}, c_d, \ldots, c_h)$ called the route for R. Let $c_{d+1} = 1 + b_d$ and assume that $c_{d+1}, c_d, \ldots, c_{i+1}$ have been specified. If $i > r$ and $R(i, q) \leq R(i-1, q)$ for some q with $c_{i+1} \leq q \leq a_i$, let c_i be the largest such q; otherwise let $c_i = a_i$ and $h = i$. Ultimately this defines h and all the c_i and hence the *route* $\pi[R]$; if $h < d$, the *shortroute* $\pi'[R]$ is the $(h + 1)$-path $\pi(c_{d+1}, c_d, \ldots, c_{h+2}, a_{h+1})$.

Now let the k-path π stand for either $\pi[R]$ or $\pi'[R]$. Let F^* consist of the nodes of F, if any, of the form $(p+t, q+t)$, with (p, q) in π and t a positive integer. Let U be the mapping on $L(s_k)$ given by $U(1, j)=R(p_j, q_j)$, where the (p_j, q_j) are the nodes of π with the standard indexing of (1.1). Clearly, U is a skew-tableau on $L(s_k)$. Let V be the mapping on $E_k=F\backslash\rho_k$ with $V(p, q)=R(p, q)$ if (p, q) is in F but is not in π or F^* and $V(p, q)=R(p+1, q+1)$ if $(p+1, q+1)$ is in F^*. We use $\sigma(R, \pi)$ to denote the pair (U, V) thus found and also let $\sigma[R]=\sigma(R, \pi[R])$ and $\sigma'[R]=\sigma(R, \pi'[R])$. The product UV is a mapping from $L(s_k)E_k$ into N with $\mu(UV)=\mu R$.

If $\sigma[R]=(U, V)$ and V is a skew-tableau on E_k, we say that R is a *k-streak*. If R is a k-streak and $\sigma'[R]=(U, V)$, it can be seen that U and V are skew-tableaux on $L(s_{k+1})$ and E_{k+1} respectively,

For example, if $F=F(\langle 6, 6, 5\rangle/\langle 4, 1\rangle)$ and the map R is given by the table

$p \backslash q$	1	2	3	4	5	6
1					1	4
2		3	5	6	8	9
3	1	2	5	7	9	

then $\pi=\{(3, 1), (3, 2), (3, 3), (2, 3), (2, 4), (2, 5), (2, 6)\}$, $\pi'=\{(3, q)|1\leq q\leq 5\}$, and $F^*=\{(3, 4), (3, 5)\}$. Also $\sigma[R]=(U, V)$ and $\sigma'[R]=(U', V')$ where U, V, U', V' are depicted respectively, by

$$U=1\ 2\ 5\ 5\ 6\ 8\ 9, \qquad V=\begin{matrix} & & 1 & 4 \\ 3 & 7 & 9 & \end{matrix},$$

$$U'=1\ 2\ 5\ 7\ 9, \qquad V'=\begin{matrix} & & & 1 & 4 \\ 3 & 5 & 6 & 8 & 9. \end{matrix}$$

7. The Reverse Correspondence

Here we show how skew-tableaux U and V on $L(s_k)$ and E_k, respectively, determine a streak $R=\tau(U, V)$ on F. We will see that (U, V) is either $\sigma[R]$ or $\sigma'[R]$. First we find a k-path π in the form $\{(p_j, q_j): 1\leq j\leq s_k\}$. The (p_j, q_j) are specified by backward induction. For $j=s_k$, let $(p_j, q_j)=(k, a_k)$. Assume that the (p_j, q_j) are known for $s_k\geq j\geq i$. If $p_i<d$ and also

$$q_i=1+b_{p_i} \quad \text{or} \quad V(p_i, q_i-1)\leq U(1, i), \tag{7.1}$$

let $(p_{i-1}, q_{i-1})=(1+p_i, q_i)$; otherwise let $(p_{i-1}, q_{i-1})=(p_i, q_i-1)$. This process is continued until one has $(p_1, q_1)=(d, 1+b_d)$. Clearly these s_k nodes form a k-path π.

Let F^* consist of the $(p+t, q+t)$ of F with (p, q) in π and t positive. Let R be the mapping on F with $R(p_j, q_j)=U(1, j)$ for the (p_j, q_j) of π, $R(p, q)=V(p-1, q-1)$ for the (p, q) of F^*, and $R(p, q)=V(p, q)$ for the remaining nodes of F. We wish to show that R is an rm-map, i.e. that $R(p, q-1)\leq R(p, q)$ whenever $(p, q-1)$ and (p, q) are in F. This is clear if these nodes are both in π or are both not in π. If (p, q) is in π and $(p, q-1)$ is not, the desired inequality follows from the definition of π, especially display (7.1). So let $(p, q-1)$ be the node (p_j, q_j) of π and (p, q) be in F^*. Then $(p_{j+1}, q_{j+1})=(p-1, q-1)$. We consider cases. If $(p-1, q)$ is in π, $(p-1, q)=(p_{j+2}, q_{j+2})$ and it follows from the definition of π that $U(1, j+2) < V(p_{j+2}, q_{j+2}-1)$; hence

$$R(p, q-1) = U(1, j) \leq U(1, j+2) < V(p_{j+2}, q_{j+2}-1)$$
$$= R(p_{j+2}+1, q_{j+2}) = R(p, q)$$

and so $R(p, q-1) \leq R(p, q)$ as desired. If $(p-1, q)$ is in F^*, an inductive argument on p allows us to assume that $R(p-1, q-1) \leq R(p-1, q)$ and so

$$R(p, q-1) = U(1, j) \leq U(1, j+1) = R(p-1, q-1) \leq R(p-1, q)$$
$$= V(p-2, q-1) < V(p-1, q-1) = R(p, q).$$

Hence R is an rm-map on F. If $R(k-1, q) < R(k, q)$ whenever (k, q) is in π and $(k-1, q)$ is in F, one sees from the algorithms for σ and τ that $\pi[R] = \pi$, $\sigma[R] = (U, V)$, and so R is a k-streak. If $R(k-1, q) \geq R(k, q)$ for some (k, q) in π, one finds similarly that $\pi[R]$ is a $(k-1)$-path, R is a $(k-1)$-streak, and $\sigma'[R] = (U, V)$.

8. The Main Results

We continue to use the notations of Sections 5–7. In particular, we recall that $\rho_d, \rho_{d-1}, \ldots, \rho_r$ are the rimpaths of F, $s_i = |\rho_i|$, and $E_i = F_i \backslash \rho_i$.

THEOREM 1. $F \sim L(s_d)E_d - L(s_{d-1})E_{d-1} + \cdots + (-1)^{d-r}L(s_r)E_r$.

PROOF. Let M be a fixed measure with $|M| = |F|$. For $r \leq i \leq d$, let S_i be the set of ordered pairs (U, V) such that U and V are skew-tableaux on $L(s_i)$ and E_i, respectively, and $\mu(UV) = M$. Let n_i and n_i' be the numbers of (U, V) in S_i such that $\tau(U, V)$ is an i-streak and $\tau(U, V)$ is an $(i-1)$-streak, respectively. Then it follows from the material in Sections 6 and 7 that $n_d = \nu(F, \varnothing, M)$, $n_r' = 0$, and

$$n_i' = n_{i-1} \quad \text{and} \quad \nu(L(s_i)E_i, \varnothing, M) = n_i + n_i' \quad \text{for } r \leq i \leq d.$$

Now the desired result follows from

$$n_d = (n_d + n_d') - (n_{d-1} + n_{d-1}') + \cdots + (-1)^{d-r-1}(n_{r+1} + n_{r+1}') + (-1)^{d-r}n_r.$$

For any finite set T of nodes, let $g(T)$ denote the number of rows containing at least one node of T. For a slicing $\pi_1, \ldots, \pi_k$ of a frame F, let the slicing *count* be

$$g(\pi_1) + \cdots + g(\pi_k) - k.$$

THEOREM 2. $F \sim \sum (-1)^c L(|\pi_1|)L(|\pi_2|) \cdots L(|\pi_k|)$, *where the sum is over all slicings* $\pi_1, \ldots, \pi_k$ *of* F *and* c *is the slicing count.*

PROOF. This follows by reapplying Theorem 1 to the E_i in the statement of Theorem 1 and continuing the process until F is equivalent to a linear combination of products of linear frames $L(s)$.

THEOREM 3. $(F, G) \sim \sum (-1)^{|G|+c}(L_1L_2 \cdots L_k, \varnothing)$, *where the sum is over all slicings* $\pi_1, \ldots, \pi_k$ *for the arena* (F, G), c *is the slicing count, and* L_i *denotes* $L(|\pi_i|)$.

The proof is by induction on the cardinality $|G|$ of the gate G. The case with $|G| = 0$ (i.e. $G = \varnothing$) is Theorem 2. So let $|G| > 0$; then some h is in G. Let A_1 be the arena (F_1, G_1) in which F_1 consists of the (p, q) of F with $p \leq h$ and $g \in G_1$ iff $g \in G$ and $g < h$.

Let $A_2=(F_2, G_2)$ with F_2 consisting of all $(p-h, q)$ such that $(p, q)\in F$ and $p>h$ and G_2 consisting of the $g-h$ such that $g\in G$ and $g>h$. Let G^* be G with h deleted.

Then one sees readily that $A_1A_2\sim(F, G)+(F, G^*)$, i.e., $(F, G)\sim A_1A_2-(F, G^*)$. Since G^* and the gate of A_1A_2 each have cardinality $|G|-1$, one can use the inductive hypothesis on A_1A_2 and on (F, G^*) and the desired result follows, using the definition of slicing for an arena.

9. Λ as a Ring of Polynomials

Introducing zero components if necessary, we let $\alpha=\langle a_1,\ldots, a_d\rangle$ and $\beta=\langle b_1,\ldots, b_d\rangle$ be any partitions. If $\alpha\neq\beta$ and the non-zero a_i-b_i with smallest i is positive, we write $\beta<\alpha$; this ordering is linear. Let M_β be the measure $\{m_n\}$ with $m_n=b_{n+1}$ for $0\leqslant n<d$ and $m_n=0$ for $n\geqslant d$. Let X_β be the product $X[M_\beta]$, as defined at the end of Section 3.

Theorem 2 tells us that integers $H_{\beta\alpha}$ exist such that

$$F(\alpha)\sim\sum H_{\beta\alpha}X_\beta, \quad \text{summed over all } \beta \text{ with } |\beta|=|\alpha|. \tag{9.1}$$

Examining the slicing process in the special case of tableau frames, one can see that $H_{\alpha\alpha}=1$ and $H_{\beta\alpha}=0$ for $\beta<\alpha$. Hence the $H_{\beta\alpha}$ for all α and β of some fixed size form a triangular matrix of integers with diagonal entries all 1, if the rows and columns are arranged by the linear ordering of partitions. Thus such a matrix has an inverse with integral entries $H'_{\alpha\beta}$. Also

$$X_\beta\sim\sum H'_{\alpha\beta}F(\alpha), \quad \text{summed over all } \alpha \text{ with } |\alpha|=|\beta|, \tag{9.2}$$

$H'_{\beta\beta}=1$, and $H'_{\alpha\beta}=0$ for $\alpha<\beta$. It follows from the authors' work in [4] that $H'_{\alpha\beta}=\nu(F(\alpha), \varnothing, M_\beta)$ and hence that the $H'_{\alpha\beta}$ are the Kostka numbers $K_{\alpha\beta}$ of the first kind defined in the study [5] of symmetric functions. Then the $H_{\beta\alpha}$ are the Kostka numbers of the second kind. Thus one application of Theorem 2 is a combinatorial algorithm for inverting the matrices of $K_{\alpha\beta}$.

One can see directly that $\nu(F(\alpha), \varnothing, M_\gamma)=0$ for $\alpha<\gamma$. Let $A=(F(\alpha), \varnothing)$, $B=(F(\beta), \varnothing),\ldots, C=(F(\gamma), \varnothing)$ with $\alpha<\beta<\cdots<\gamma$. Also let $D=aA+bB+\cdots+cC$ with $a, b,\ldots, c$ integers. Then $\nu[D, M_\gamma]=c$. It follows that $\nu[D, M]=0$ for all measures M only if $0=c=\cdots=b=a$. Thus, in Λ the tableau frames $F(\alpha)$, as arenas $(F(\alpha), \varnothing)$, are linearly independent over the integers Z.

Theorem 3 shows that the products X_β, for all partitions β, generate Λ as a module over Z. Then (9.2) shows that the tableau frames $F(\alpha)$ also generate Λ. Since the $F(\alpha)$ are linearly independent, they form a basis for the module Λ. As in [4], one sees by a simple combinatorial argument that the X_β also form a basis. This means that the $x_s=(L(s), \varnothing)$ are algebraically independent indeterminates over Z and Λ is the polynomial ring $Z[x_1, x_2,\ldots]$.

10. Schur Functions and other Enumerators

Let $z_0, z_1,\ldots$ be an infinite sequence of independent indeterminates over Z. If $M=\{m_n\}$ is a measure and $i, j,\ldots, k$ are the ns for which $m_n>0$, let $Z(M)$ be the power product

$$(z_i)^{m_i}(z_j)^{m_j}\cdots(z_k)^{m_k}.$$

For C in Λ, we define the *Schur function* $S[C]$ to be the formal infinite sum $\sum_M\nu[C, M]Z(M)$, where $\sum_M$ denotes a sum over all measures M. If one wishes to avoid formal infinite sums, $S[C]$ could be defined as the mapping from the set of all measures into N with $M\mapsto\nu[C, M]$. The classical partition Schur function $\{\alpha\}$ and skew Schur function $\{\alpha/\beta\}$ are the special cases $S[F(\alpha)]$ and $S[F(\alpha/\beta)]$, respectively.

Let z be an indeterminate over Z. For a fixed arena $A=(F, G)$ and each w in N let $g_w(A)$ be the number of maps R on A with weight $\|R\|=\sum R(p,q)=w$; then the *enumerator* of A (or generating function for A) is

$$\Gamma[A]=g_0(A)+g_1(A)z+g_2(A)z^2+\cdots.$$

For m and w in N, let $g_{mw}(A)$ be the number of maps R on A with $\|R\|=w$ and each entry $R(p,q)\leq m$ and let

$$\Gamma_m[A]=g_{m0}(A)+g_{m1}(A)z+g_{m2}(A)z^2+\cdots.$$

For $x_s=(L(s), \varnothing)$, it is well known that

$$\Gamma[x_s]=[(1-z)(1-z^2)\cdots(1-z^s)]^{-1},$$

$$\Gamma_m[x_s]=\left[\prod_{i=1}^{m}(1-z^{s+i})\right]\div\left[\prod_{i=1}^{m}(1-z^i)\right]. \tag{10.1}$$

Using $\Gamma(\sum u_iA_i)=\sum u_i\Gamma(A_i)$, one extends Γ to Ω and hence to Λ; the same is done with Γ_m. Multiplication in Ω and Λ is such that

$$S[CC']=S[C]S[C'], \qquad \Gamma[CC']=\Gamma[C]\Gamma[C'], \qquad \Gamma_m[CC']=\Gamma_m[C]\Gamma_m[C'],$$

where multiplication of Schur functions and of enumerators is the Cauchy product of formal infinite series. Thus the mappings S, Γ and Γ_m are ring homomorphisms from either Ω or Λ into suitable rings. Using the formulas of (10.1) and Theorem 3, one can find $\Gamma[C]$ and $\Gamma_m[C]$ for any C in Ω. The use of Theorem 2 to find a skew Schur function is illustrated in Section 12.

Let us now consider each of these three homomorphisms to have Λ as its domain. Our definition of equivalence is such that S is injective. However, the others are not. For example, $F(\langle 3\rangle)-F(\langle 1,1,1\rangle)-F(\langle 2\rangle)$ is in the kernel of Γ. Also, the images under Γ_1 of the seven arenas

$$x_5,\ x_4x_1,\ x_3x_2,\ x_3x_1^2,\ x_2^2x_1,\ x_2x_1^3,\ x_1^5$$

are all polynomials in z of degree 5, hence these seven images are linearly dependent over the integers and the kernel of Γ_1 is not trivial. Similarly one shows for all positive integers m that Γ_m is not injective.

11. The Dual Process

The concepts, procedures and theorems in previous sections all have duals resulting from the interchange of rows and columns and of $<$ and $\leq$. We state some of these duals here.

The dual overlap for a frame F consists of the column numbers q for which there exist p with (p, q) and $(p, q+1)$ in F. A dual gate G^* for F is a subset of the dual overlap for F and a dual arena is an ordered pair $[F, G^*]$, with G^* a dual gate for F. A mapping R from F into N is a column strict map (cs-map) on F if $R(p,q)$ is a strictly increasing function of p for fixed q. The dual gate γ^*R is the set of all q for which (p, q) and $(p, q+1)$ exist in F with $R(p,q)>R(p,q+1)$. R is a dual map on a dual arena $[F, G^*]$ if R is a cs-map on F and $\gamma^*R=G^*$. Thus a dual map on $[F, \varnothing]$ is a skew-tableau on F. If $A^*=[F, G^*]$ is a dual arena and M is a measure, $\nu^*[A^*, M]$ denotes the number of dual maps R on A^* with $\mu R=M$. The definitions of Ω^* and Λ^* are clear.

The transpose $(F, G)^{\mathrm{T}}$ of an arena (F, G) is the dual arena $[F^{\mathrm{T}}, G]$. A dual h-path in F is the transpose of an h-path in F^{T}. Similarly, one defines dual rimpaths ρ_h^*, dual complements $E_h^*=F\backslash\rho_h^*$, and dual slicings for a frame F and for dual arenas (F, G^*).

Then the dual of Theorem 1 is the following:

THEOREM 1*. *In F, let dual rimpaths ρ_q^* exist only for $r \leqslant q \leqslant d$ and let $|\rho_q^*| = s_q$. Then in Λ^*,*

$$F = [L(s_d)]^{\mathrm{T}}E_d^* - [L(s_{d-1})]^{\mathrm{T}}E_{d-1}^* + \cdots + (-1)^{d-r}[L(s_r)]^{\mathrm{T}}E_r^*.$$

Similarly one has duals of Theorems 2 and 3. It follows from Theorem 3* that $\Lambda^* = Z[y_1, y_2, \ldots]$; hence $\Lambda^* = \Lambda$. The mapping with $(F, G) \mapsto (F, G)^{\mathrm{T}}$ extends into the conjugation automorphism of Λ (described in [4]).

THEOREM 4. *Let $F = F(\langle a_1, \ldots, a_d\rangle/\langle b_1, \ldots, b_d\rangle)$ and $x_{ij} = L(a_i - i - b_j + j)$, where $L(m) = 0$ for $m = -1, -2, \ldots$. Then F is equivalent to the d by d determinant* $\det(x_{ij})$.

The proof follows by induction on d using Theorem 1. We also note that the proof of Theorem 4 (or its dual) follows from the isomorphism $C \mapsto S[C]$ of Section 10, and the classical expression for a skew Schur function as a determinant of homogeneous (or of elementary) symmetric functions. The determinant expressions for $S[F(\alpha)]$ are due to Jacobi, Trudi and Naegelsbach while those for the more general $S[F(\alpha/\beta)]$ arise from a theorem of Aitkin. (See displays (6.3; 2), (6.3; 3), VIII and IX on pages 88, 89, 110 and 110, respectively, in Littlewood [6].)

12. EXAMPLES

First let $F = F(\langle 5, 5, 3\rangle/\langle 4, 1\rangle)$. Since $a_3 = 3 > 1 = b_2$ and $a_2 = 5 > 4 = b_1$, the rimpaths ρ_h exist for $h \in \{1, 2, 3\}$. One sees that

$$\rho_3 = \{(3,1), (3,2), (3,3)\}, \qquad \rho_2 = \rho_3 \cup \{(2,3), (2,4), (2,5)\},$$
$$\rho_1 = \rho_2 \cup \{(1,5)\}, \qquad |\rho_3| = 3, |\rho_2| = 6, |\rho_1| = 7,$$
$$E_3 = F(\langle 5, 5\rangle/\langle 4, 1\rangle), \qquad E_2 = \{(2,2), (1,5)\}, \qquad E_1 = \{(2,2)\}.$$

Thus Theorem 1 tells us that $F \sim x_3E_3 - x_6E_2 + x_7E_1$. The slicings for $F = (F, \varnothing)$ are

$$\pi_{11} = \rho_3, \qquad \pi_{12} = \{(2,2), (2,3), (2,4), (2,5)\}, \pi_{13} = \{(1,5)\} \qquad (S_1)$$
$$\pi_{21} = \rho_3, \qquad \pi_{22} = \pi_{12} \cup \{(1,5)\} \qquad (S_2)$$
$$\pi_{31} = \rho_2, \qquad \pi_{32} = \{(2,2)\}, \pi_{33} = \{(1,5)\}. \qquad (S_3)$$
$$\pi_{41} = \rho_1, \qquad \pi_{42} = \{(2,2)\} \qquad (S_4)$$

Then Theorem 2 gives us $F \sim x_3x_4x_1 - x_3x_5 - x_6x_1^2 + x_7x_1$. The non-null gates for F are $G_1 = \{1\}$, $G_2 = \{2\}$, and $G_3 = \{1, 2\}$. The only slicings for (F, G_1) are (S_2) and (S_4), hence $(F, G_1) \sim x_3x_5 - x_7x_1$, using Theorem 3. Similarly, $(F, G_2) \sim x_6x_1^2 - x_7x_1$ and $(F, G_3) \sim x_7x_1$.

Next we illustrate the use of Theorem 2 to express a skew Schur function as a polynomial in the elementary symmetric Schur functions. Here let $F = F(\langle 3, 3, 3, 2\rangle/\langle 2, 2, 1\rangle)$. Then the transpose $F^{\mathrm{T}} = F(\langle 4, 4, 3\rangle/\langle 3, 2\rangle)$. Applying Theorem 2, one finds that $F^{\mathrm{T}} \sim x_3x_2x_1 - x_3^2 - x_5x_1 + x_6$. Then the automorphism of conjugation in Λ gives

$$F = (F^{\mathrm{T}})^{\mathrm{T}} \sim y_3y_2y_1 - y_3^2 - y_5y_1 + y_6$$

where $y_s = ([L(s)]^{\mathrm{T}}, \varnothing)$. The Schur function $S[y_s]$ is the sum of all products of s distinct

z_i. Hence

$$S[F]=(\sum z_a z_b z_c)(\sum z_a z_b)(\sum z_a)-(\sum z_a z_b z_c)^2 \\ -(\sum z_a z_b z_c z_d z_e)(\sum z_a)+\sum z_a z_b z_c z_d z_e z_f.$$

Alternatively, one can use the definition of skew Schur function and obtain

$$S[F]=35\sum z_a z_b z_c z_d z_e z_f+15\sum z_a^2 z_b z_c z_d z_e+6\sum z_a^2 z_b^2 z_c z_d \\ +2\sum z_a^2 z_b^2 z_c^2+3\sum z_a^3 z_b z_c z_d+\sum z_a^3 z_b^2 z_c.$$

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